COMBINATORICA 2(3) (1982) 229—235

THE EDMONDS—GALLAI DECOMPOSITION FOR MATCHINGS IN LOCALLY FINITE GRAPHS

François BRY and Michel LAS VERGNAS*

Dedicated to Tibor Gallai on his seventieth birthday

Received 8 February 1982

We show that the Edmonds—Gallai decomposition theorem for matchings in finite graphs generalizes to all locally finite graphs.

1. Introduction

For a finite graph G with vertex set V the Edmonds—Gallai decomposition theorem for matchings is the following statement:

Let P be set of vertices of G not covered by all maximal matchings of G, Q the set of vertices in $V \setminus P$ adjacent to P, and $R = V \setminus (P \cup Q)$.

Then (i) *Every connected component of the subgraph $G[P]$ of G induced by P is factor-critical, and $G[R]$ is factorizable,* (ii) *every maximal matching of G is a union of a near-1-factor of each component of $G[P]$, a matching from Q into P and a* 1*-factor of $G[R]$.*

The properties (i) and (ii) immediately imply that the maximal cardinality of a matching of G is $\frac{1}{2}(|V| - c_1(P) + |Q|)$, where $c_1(P)$ is the number of odd components of $G[P]$.

This theorem, implicit in [4] and [6], is quoted explicitly in [8].

Our purpose in the present paper is to show that the Edmonds—Gallai decomposition generalizes to locally finite graphs. Our proof yields a short derivation of the Edmonds—Gallai theorem from Tutte's 1-Factor Theorem [13] in the finite case. The results of this paper are announced in [3].

2. Terminology and notations

Loops, multiple edges and edge directions being irrelevant here, we may consider that the edges of a graph $G = (V, E)$ are unordered pairs of vertices: $E \subseteq \binom{V}{2}$.

AMS subject classification (1980): 05 C 99.

* C.N.R.S.

The graph G is *finite* if V (hence also E) is finite; G is *locally finite* if every vertex is contained in a finite number of edges. A connected locally finite graph is denumerable.

A *matching* of G is a set of pairwise disjoint edges. A *perfect matching* or *1-factor* of G is a matching covering all vertices of G (i.e. every vertex of G belongs to some edge of the matching). A graph having a 1-factor is said *1-factorizable,* or more briefly *factorizable.* A graph is *factor-critical* if it is not factorizable but all induced subgraphs obtained by deleting one vertex are factorizable. A factor-critical graph is necessarily connected. We say that a matching covering all vertices of a graph except one is a *near-1-factor.*

Given a subset X of V, we denote by $G[X]$ the *subgraph of G induced by X*: $G[X]=\left(X; E\cap\binom{X}{2}\right)$. We say that a set $C\subseteq V$ is an *odd (even, infinite) component* of G if $G[C]$ is a connected component of G and $|C|$ is odd (even, infinite). We denote by $c_1(G)$ resp. $c_{\text{f. cr.}}(G)$ the (finite or infinite) number of odd resp. factor-critical components of G. We usually abbreviate $c_1(G[X])$ by $c_1(X)$ resp. $c_{\text{f. cr.}}(G[X])$ by $c_{\text{f. cr.}}(X)$ for $X\subseteq V$.

We recall Tutte's 1-Factor Theorem for locally finite graphs [14]:

A locally finite graph G with vertex-set V is factorizable if and only if $c_1(V\setminus X)\leq$ $\leq|X|$ for all finite subsets X of V.

This theorem implies in particular that a locally finite factor-critical graph is finite (with an odd number of vertices).

We denote by $V(M)$ the set of vertices covered by a matching M, $V(M)$ is the *support* of M. Edmonds and Fulkerson have shown in [5] that the subsets of matching supports of a finite graph G are the independent sets of a matroid on V — the *matching matroid* of G ([15]). This property generalizes to locally finite graphs [2]. It follows from Rado's Selection Principle that in this case the matching matroid is finitary [2] (i.e. a set X is independent if and only if every finite subset of X is independent). We say that a matching M of a locally finite graph G is *maximal* if $V(M)$ is maximal with respect to inclusion among matching supports of G (i. e. $V(M)$ is a basis of the matching matroid of G). If G is finite it follows from the Edmonds—Fulkerson theorem that all maximal matchings of G have the same cardinality (which is the maximal cardinality of a matching of G). If G is locally finite, every set X covered by a matching of G is contained in a basis of the matching matroid.

The *defect* $\delta(M)$ of a matching M of a graph G is the number of vertices not covered by M. It follows from properties of finitary matroids that

Lemma 2.1. *If some maximal matching M of a locally finite graph G has a finite defect, then a matching M' of G is maximal if and only if $\delta(M')=\delta(M)$.*

A direct proof of this property is by alternating chain methods. Let M, M' be two matchings of G. We define an *(M, M')-component* of G as a connected component of the graph $(V, (M\setminus M')\cup(M'\setminus M))$ with at least one edge. An (M, M')-component of G is an alternating cycle or an alternating chain (finite, 1-way infinite or 2-way infinite). If M and M' are both maximal then the (M, M')-components are alternating cycles or 2-way infinite chains contained in $G[V(M)\cap V(M')]$ or finite alternating chains whose ends establish a bijection between $V(M)\setminus V(M')$ and $V(M')\setminus V(M)$.

The *matching defect* $\delta(G)$ of a locally finite graph G is the defect of any maximal matching of G.

3. The matching defect of a locally finite graph

We derive in this section an expression for the matching defect $\delta(G)$ of a locally finite graph G.

Lemma 3.1. *Let G be a graph with vertex-set V, and S be a finite subset of V such that $c_1(V\setminus S)$ is finite. Then there is a finite subset of V such that $S\subseteq T\subseteq V$ and $c_{\text{f.rc.}}(V\setminus T)-|T|\geqq c_1(V\setminus S)-|S|$.*

Proof. Let $C_1, C_2, \ldots, C_p$ be the odd components of $G[V\setminus S]$. Let T be maximal with respect to inclusion with the properties $S\subseteq T\subseteq S\cup C_1\cup C_2\cup\ldots\cup C_p$ and $c_1(V\setminus T)-|T|\geqq c_1(V\setminus S)-|S|$. We prove that every odd component of $G[V\setminus T]$ is factor-critical.

Suppose there is an odd component C of $G[V\setminus T]$ which is not factor-critical. Then by Tutte's theorem (finite case) there are $x\in C$ and $X\subseteq C\setminus\{x\}$ such that $c_1(C\setminus X\setminus\{x\})\geqq|X|+1$. Since $|C|$ is odd $c_1(C\setminus X\setminus\{x\})$ has the parity of $|X|$, hence $c_1(C\setminus X\setminus\{x\})\geqq|X|+2$. We have $c_1(V\setminus(T\cup X\cup\{x\})=c_1(V\setminus T)-1+c_1(C\setminus X\setminus\{x\})$. It follows $c_1(V\setminus(T\cup X)\cup\{x\})-|T\cup X\cup\{x\}|>c_1(V\setminus T)-|T|$, contradicting the choice of T. ∎

Proposition 3.2. *Let G be a locally finite graph with vertex-set V. The matching defect of G is given by*

$$\delta(G)=\max_{\substack{S\subseteq V\\ S\text{ finite}}}\big(c_1(V\setminus S)-|S|\big)=\max_{\substack{S\subseteq V\\ S\text{ finite}}}\big(c_{\text{f.cr.}}(V\setminus S)-|S|\big).$$

In the finite case the first formula is given by Berge in [1] (see also [11]). A standard proof as a corollary of Tutte's theorem follows from the observation that $\delta(G)$ is the least number of new vertices adjacent to all vertices of G one has to add so that the resulting graph has a 1-factor. This idea cannot be used here, since the auxiliary graph would not be locally finite.

Proof. Clearly $\delta(G)\geqq c_1(V\setminus S)-|S|$ for any finite $S\subseteq V$, since at most $|S|$ odd components of $G[V\setminus S]$ can be covered by any matching of G. Hence if $c_1(V\setminus S)-|S|$ is not bounded, $\delta(G)$ is infinite.

Suppose $c_1(V\setminus S)-|S|$ is bounded and consider a finite $S\subseteq V$ achieving its maximum. By Lemma 3.1. we may suppose that every odd component of $G[V\setminus S]$ is factor-critical. Let $T\subseteq S$. By the choice of S we have $c_1\big(V\setminus(S\setminus T)\big)-|S\setminus T)|\leqq c_1(V\setminus S)-|S|$. Hence $c_1(V\setminus S)-c_1\big(V\setminus(S\setminus T)\big)\geqq|T|$, implying that T is adjacent to at least $|T|$ odd components of $G[V\setminus S]$. It follows then from the König—Hall theorem that there is a matching M_0 from S into $V\setminus S$ meeting $|S|$ odd components of $G[V\setminus S]$. Each odd component C of $G[V\setminus S]$ being factor-critical contains a near-1-factor M_C which does not meet M_0.

Consider now an even or infinite component C of $G[V\setminus S]$. Let $T\subseteq C$. By the choice of S we have $c_1(V\setminus S)+c_1(C\setminus T)-|S\cup T|=c_1\big(V\setminus(S\cup T)\big)-|S\cup T|\leqq c_1(V\setminus S)-|S|$, hence $c_1(C\setminus T)\leqq|T|$. By Tutte's theorem $G[C]$ is factorizable: let M_C be a 1-factor.

Then $M=M_0\cup\bigcup M_C$, where the union is over all components C of $G[V\setminus S]$, is a matching of G with defect $c_1(V\setminus S)-|S|$, proving the proposition. ∎

In the finite case parts of the above proofs appear in several places of the literature: see for instance [10] proof of Satz 2.1, [12].

Remark 3.3. An alternative proof of Proposition 3.2 can be derived from an expression for the rank function of the matching matroid of a locally finite graph due to Brualdi [2] (see also [7], [9] for the finite case): Given a finite $X \subseteqq V$, let $r(X)$ be the maximal number of vertices of X which can be covered by a matching of G ($r(X)$ is the rank of X in the matching matroid of G). We have

$$r(X) = \min_{\substack{S \subseteqq V \\ S \text{ finite}}} (|X| + |S| - c_1(V \setminus S; X))$$

where $c_1(V \setminus S; X)$ denotes the number of odd components of $G[V \setminus S]$ contained in X.

The graph G being locally finite it follows from Rado's Selection Principle that $\delta(G) = \max_{X \text{ finite} \subseteqq V} (|X| - r(X))$. This statement generalizes [2] Theorem 4. The proof is similar and left to the reader. Hence, by Brualdi's theorem we have

$$\delta(G) = \max_{X \text{ finite} \subseteqq V} \max_{S \text{ finite} \subseteqq V} (c_1(V \setminus S; X) - |S|).$$

If $c_1(V \setminus S)$ is infinite for some finite S then $\delta(G)$ is infinite. If $c_1(V \setminus S)$ is finite then for a finite X containing the odd components of $G[V \setminus S)$ we have $c_1(V \setminus S;\ X) = c_1(V \setminus S)$. Hence $\delta(G) = \operatorname{Max}_{S \text{ finite} \subseteqq V} (c_1(V \setminus S) - |S|)$.

The second formula follows from Lemma 3.1.

Remark 3.4. It follows from Lemma 3.1 that odd components may be replaced by factor-critical components in several statements relative to matchings.

For instance from Tutte's 1-Factor Theorem for locally finite graphs we get that a locally finite graph G with vertex-set V has a 1-factor if and only if $c_{\text{f.cr.}}(V \setminus S) \leqq |S|$ for all finite $S \subseteqq V$ (also an immediate corollary from Proposition 3.2). In the finite case this theorem is well-known, as an immediate consequence of the Edmonds — Gallai theorem.

Similarly Brualdi's theorem can be stated: for any finite $X \subseteqq V$

$$r(X) = \min_{\substack{S \subseteqq V \\ S \text{ finite}}} (|X| + |S| - c_{\text{f.cr.}}(V \setminus S; X))$$

where $c_{\text{f.cr.}}(V \setminus S; X)$ is the number of factor-critical components of $G[V \setminus S]$ contained in X (for a proof use the immediate extension of Lemma 3.2 with $c_{\text{f.cr.}}(V \setminus S; X)$ instead of $c_{\text{f.cr.}}(V \setminus S)$).

4. The Edmonds—Gallai decomposition for locally finite graphs

We first prove the decomposition theorem in the finite defect case.

Theorem 4.1. *Let G be a locally finite graph with finite matching defect. Let V be the vertex-set of G, P be the set of vertices not covered by all maximal matchings of G, and Q be the set of vertices in $V \setminus P$ adjacent to P. We set $R = V \setminus (P \cup Q)$.*

Then (i) *P is finite (hence also Q), all the connected components of $G[P]$ are factor-critical, $G[R]$ is factorizable,* (ii) *every maximal matching of G is a union of a near-*1*-factor of each connected component of $G[P]$, a matching from Q into P and a* 1*-factor of $G[R]$.*

Clearly it follows from properties (i) and (ii) that the matching defect of G is $c_1(P)-|Q|$.

Proof. Let S be a finite subset of V such that $\delta(G)=c_{\text{f.cr.}}(V\setminus S)-|S|$ (cf. Proposition 3.2). Note that necessarily $c_1(V\setminus S)=c_{\text{f.cr.}}(V\setminus S)$. Let T be a subset of S such that T is adjacent to at most $|T|$ odd components of $G[V\setminus S]$ and T is maximal with respect to inclusion with this property (such a T exists since $\emptyset$ has the considered property and S is finite).

Let P' be the union of all odd components of $G[V\setminus S]$ not adjacent to T, $Q'=S\setminus T$ and $R'=V\setminus(P'\cup Q')$. We show that $P=P'$ and $Q=Q'$.

Since all the vertices adjacent to P' are in Q' we have $\delta(G)\geqq c_1(P')-|Q|'$. Now $c_1(P')=c_1(V\setminus S)-k$, where k is the number of odd components of $G[V\setminus S]$ adjacent to T. Hence $\delta(G)\geqq c_1(P')-|Q'|=c_1(V\setminus S)-k-|Q'|\geqq c_1(V\setminus S)-|T|-|S\setminus T|=\delta(G)$. This implies $\delta(G)=c_1(P')-|Q'|$ and T is adjacent to exactly $|T|$ odd components of $G[V\setminus S]$. By Lemma 2.1 any maximal matching M of G has defect $\delta(M)=\delta(G)=c_1(P')-|Q'|$. Therefore, all vertices adjacent to P' being in Q', M necessarily contains a near-1-factor of each component of $G[P']$, a matching from Q' into P' (meeting Q' components of $G[P']$) and a 1-factor of $G[R']$. It follows that $P\subseteqq P'$ and Q' is exactly the set of vertices adjacent to P'.

On the other hand any non-empty subset X of Q' is adjacent to at least $|X|+1$ components of $G[P']$ (otherwise T would not be maximal). It follows from the König—Hall theorem that for any given component C of $G[P']$ there is a matching M_2 from Q' into P' meeting $|Q'|$ components of $G[P']$ different from C. Since the components of $G[P']$ are factor-critical, given any $x\in C$ there is a matching M_1 of $G[P']$ not meeting M_2 and not containing x which is a union of near-1-factors of each components of $G[P']$. Set $M=M_1\cup M_2\cup M_3$, where M_3 is a 1-factor of $G[R']$. Then M is a matching of G with defect $c_1(P')-|Q'|=\delta(G)$ which does not contain x. Hence $P'\subseteqq P$. It follows that $P=P'$ and $Q=Q'$, proving both (i) and (ii). ∎

We now prove the general decomposition theorem.

Theorem 4.2. *Let G be a locally finite graph with vertex-set V. Let P be the set of vertices of G not covered by all maximal matchings of G, Q be the set of vertices in $V\setminus P$ adjacent to P and $R=V\setminus(P\cup Q)$.*

Then (i) *every connected component of $G[P]$ is factor-critical and $G[R]$ is factorizable,* (ii) *every maximal matching of G is a union of a near-*1*-factor of each component of $G[P]$, a matching from Q into P and a* 1*-factor of $G[R]$.*

Proof. Let M_0 be a maximal matching of G with support V_0. We may suppose $\delta(G)=|V\setminus V_0|$ infinite since the finite case is given by Theorem 4.1. It suffices to consider the case when G is connected. Then V is denumerable. We set $V\setminus V_0=\{x_1, x_2, \ldots\}$, $V_i=V_0\cup\{x_1, x_2, \ldots, x_i\}$ and $G_i=G[V_i]$, $i=0, 1, \ldots$.

(1) Let P_i be the set of vertices of V_i not covered by all maximal matchings of G_i. We observe that M_i is maximal in G_i with defect i. Hence by Lemma 2.1 a matching is maximal in G_i if and only if its defect is i. It follows that a matching maximal in G_i is maximal in G_{i+1}, hence $P_i\subseteqq P_{i+1}$. We set $P_\infty=\bigcup_{i\geqq 0} P_i$. We show that $P_\infty=P$.

A matching M maximal in G_i is also maximal in G. Suppose for a contradiction that there is a matching N of G such that $V(M)\subset V(N)$. Let M' be the matching obtained from M by exchanging M-edges and N-edges in the (M, N)-component of some $x \in V(N) \setminus V(M)$. For $j \geqq i$ large enough V_j contains the ends of this (M, N). component. Then M' is a matching of G_j with $V(M) \cup \{x\} \subseteqq V(M')$, a contradiction-since M is maximal in G_j by the above remark. It follows that $P_i \subseteqq P$ for all i, hence $P_\infty \subseteqq P$.

To prove the reverse inclusion we show that $V \setminus V(M) \subseteqq P_\infty$ for any maximal matching M of G. We have $x_1, x_2, \ldots \in P_\infty$. Consider $z \in V_0 \setminus V(M)$. Since M_0 and M are both maximal matchings of G, there is a bijection between $V_0 \setminus V(M)$ and $V(M) \setminus V_0$ by (M_0, M)-alternating (finite) chains. Let $x_i \in V(M) \setminus V_0$ be the vertex associated with z by this bijection. Exchanging M_0-edges and M-edges on the corresponding chain we obtain from M_0 a maximal matching of G_i with support $(V_0 \setminus \{z\}) \cup \{x_i\}$. Hence $z \in P_i$.

(2) Let Q_i be the set of vertices of $V_i \setminus P_i$ adjacent to P_i. We show that $Q_i \subseteqq Q_{i+1}$.

Since $P_i \subseteqq P_{i+1}$ we have $Q_i \subseteqq P_{i+1} \cup Q_{i+1}$. Suppose there is $y \in Q_i \cap P_{i+1}$. Let M be a maximal matching of G_{i+1} such that $y \in V(M)$. We observe that x_{i+1} is not adjacent to P_i. Otherwise by extending in the obvious way a maximal matching of G_i not covering some vertex of P_i adjacent to x_{i+1} we get a matching of defect $< i+1$ in G_{i+1}. Thus the set of neighbours of P_i in G_{i+1} is Q_i. Since M covers at most $|Q_i|-1$ vertices of Q_i it covers at most $|Q_i|-1$ odd components of $G[P_i]$. Hence the defect of M in G_{i+1} is $\geqq c_1(P_i) - (|Q_i|-1) + 1 = i+2$, a contradiction.

We set $Q_\infty = \bigcup_{i \geqq 0} Q_i$. We show that $Q_\infty = Q$. Let $x \in Q_i$. Since $Q_i \subseteqq Q_{i+1} \subseteqq \ldots$ x is not in any P_j hence $x \notin P$. It follows that $x \in Q$ showing that $Q_\infty \subseteqq Q$. Conversely let $x \in Q$. Since x is adjacent to P, x is adjacent to P_i for some i. Now $x \in V \setminus P = V \setminus \bigcup_{i \geqq 0} P_i$, hence $x \in Q_i$.

(3) Since $x \in Q_i$ implies $x \notin P$ the connected components of $G[P_i]$ are components of $G[P]$. Since $P = \bigcup_{i \geqq 0} P_i$ it follows that any component of $G[P]$ is a component of $G[P_i]$ for some large enough. i. Hence the connected components of $G[P]$ are factor-critical by Theorem 4.1.

(4) Let M be a maximal matching of G.

(4a) There is at most one vertex not covered by M in any connected component of $G[P]$.

Suppose there are $x, y \in C \setminus V(M)$, where C is a component of $G[P]$. Consider the (M_0, M)-components containing x and y. These components are two distinct chains otherwise M would not be maximal; they are contained in V_i for some i large enough, since all their inner vertices are in V_0. Then the matching obtained from M_0 by exchanging M_0-edges and M-edges on these chains is a maximal matching of G_i which does not cover x and y, contradicting Theorem 4.1.

(4b) There is at most one edge of M meeting a connected component C of $G[P]$ and not contained in this component.

Suppose for a contradiction that there are two edges e_1, e_2 in M joining C to Q. As above, the matching obtained from M_0 by exchanging M_0-edges and M-edges in the (M_0, M)-components of e_1 and e_2 is a maximal matching of G_i for a large enough i. This matching contains e_1 and e_2 contradicting Theorem 4.1.

(4c) Every edge of M incident to Q has its other end in P. The proof along the lines of (4a), (4b) is left to the reader.

In the case of finite defect, we get back Theorem 4.1 from Theorem 4.2. It suffices to show that if $\delta(G)$ is finite then P is also finite. This follows from a lemma on finitary matroids ($Q \cup R$ is the set of isthmuses of the matching matroid).

Lemma 4.3. *Let M be a finitary matroid on a set E. If there is a basis B of M such that $E \setminus B$ is finite then the set of non-isthmuses of M is finite.*

Proof. The set of non-isthmuses of M is the set $P = \bigcup_{e \in E \setminus B} C_e$, where C_e denotes the unique circuit of M contained in $B \cup \{e\}$ ([15]). Since M is finitary each C_e is finite, hence P is finite if $E \setminus B$ is finite for some basis B (and then $E \setminus B$ is finite for every basis B). ∎

References

[1] C. Berge, Sur le couplage maximum d'un graphe, *C. R. Acad. Sci. Paris*, **247** (1958) 258—259.
[2] R. A. Brualdi, Matchings in arbitrary graphs, *Proc. Camb. Phil. Soc.*, **69** (1971) 401—407.
[3] F. Bry, Sur les couplages dans les graphes localement finis, Coll. Int. sur la Théorie des Graphes et la Combinatoire Marseille-Luminy 1981, *Ann. Discrete Math., to appear.*
[4] J. Edmonds, Paths, trees and flowers, *Canad. J. Math.*, **17** (1965) 449—467.
[5] J. Edmonds and D. R. Fulkerson, Transversals and matroid partition *J. Res. Nat. Bur. Stand.*, ser. B, **69** (1965) 147—153.
[6] T. Gallai, Maximale Systeme unabhängiger Kanten, *Mat. Kut. Int. Közl.*, **9** (1964) 373—395.
[7] M. Las Vergnas, Degree-constrained subgraphs and matroid theory, *in: Infinite and Finite Sets, Proc. Colloq. Math. Soc. J. Bolyai* **10**, (A. Hajnal, R. Rado and V. T. Sós, eds) Keszthely (Hungary) 1973, North-Holland 1976, pp. 1473—1502.
[8] L. Lovász, On the structure of factorizable graphs, *Acta Math. Acad. Sci. Hung.*, **23** (1972) 179—185.
[9] P. J. McCarthy, Matchings in graphs II, *Discrete Math.* **11** (1975) 141—145.
[10] W. Mader, 1-Faktoren von Graphen, *Math. Ann.*, **201** (1973) 269—282.
[11] O. Ore, Graphs and subgraphs, *Trans. Amer. Math. Soc.*, **84** (1957) 109—137.
[12] J. Plesnik, Connectivity of regular graphs and the existence of 1-factor, *Mat. Casopis*, **22** (1972) 310—318.
[13] W. T. Tutte, The factorization of linear graphs, *J. London Math. Soc.*, **22** (1947) 107—111.
[14] W. T. Tutte, The factorization of locally finite graphs, *Canad. J. Math.*, **2** (1950) 44—49.
[15] D. J. A. Welsh, *Matroid Theory*, Academic Press, 1976.

François Bry
Michel Las Vergnas

Université Pierre et Marie Curie
U.E.R. 48
4 place Jussieu,
75005 Paris, France